Introduction to Linear and Nonlinear Programming

Outline

- Introduction
  - types of problems
  - size of problems
  - iterative algorithms and convergence
- Basic properties of Linear Programs
  - introduction
  - examples of linear programming problems

Types of Problems

- Three parts
  - linear programming
  - unconstrained problems
  - constrained problems
- The last two parts comprise the subject of nonlinear programming

Linear Programming

- It is characterized by linear functions of the unknowns; the objective is linear in the unknowns, and the constraints are linear equalities or linear inequalities.
- Why are linear forms for objectives and constraints so popular in problem formulation?

  - a great number of constraints and objectives that arise in practice are indisputably linear (ex: budget constraint)
  - they are often the least difficult to define

Unconstrained Problems

- Are unconstrained problems devoid of structural properties as to preclude their applicability as useful models of meaningful problems?
  - if the scope of a problem is broadened to the consideration of all relevant decision variables, there may then be no constraints
  - many constrained problems are sometimes easily converted to unconstrained problems

Constrained Problems

- Many complex problems cannot be directly treated in its entirety accounting for all possible choices, but instead must be decomposed into separate subproblems
- "*Continuous variable programming*"

Size of Problems

- Three classes of problems
  - small scale: five or fewer unknowns and constraints
  - intermediate scale: from five to a hundred variables
  - large scale: from a hundred to thousands variables

Iterative Algorithms and Convergence

- Most algorithms designed to solve large optimization problems are iterative.
- For LP problems, the generated sequence is of finite length, reaching the solution point exactly after a finite number of steps.
- For non-LP problems, the sequence generally does not ever exactly reach the solution point, but converges toward it.

Iterative Algorithms and Convergence (cont'd)

- Iterative algorithms
  1. the creation of the algorithms themselves
  2. the verification that a given algorithm will in fact generate a sequence that converges to a solution
  3. the rate at which the generated sequence of points converges to the solution
- Convergence rate theory
  1. the theory is, for the most part, extremely simple in nature
  2. a large class of seemingly distinct algorithms turns out to have a common convergence rate

LP Problems' Standard Form

minimize $\quad c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots c_n x_n$

subject to $\quad a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots a_{2n}x_n = b_2$$

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$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots a_{mn}x_n = b_m$$

and $\quad x_1 \geq 0, x_2 \geq 0, \ldots\ldots\ldots x_n \geq 0$

LP Problems' Standard Form (cont'd)

$$\text{minimize} \quad c^T x$$

$$\text{subject to} \quad Ax=b \qquad \text{and } x \geq 0$$

- Here x is an n-dimensional column vector, c^T is an n-dimensional row vector, A is an m×n matrix, and b is an m-dimensional column vector.

Example 1 (Slack Variables)

$$\text{minimize} \quad c_1 x_1 + c_2 x_2 + \text{..........} c_n x_n$$

$$\text{subject to} \quad a_{11} x_1 + a_{12} x_2 + \text{...........} a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \text{...........} a_{2n} x_n \leq b_2$$

$$.$$

$$.$$

$$.$$

$$a_{m1} x_1 + a_{m2} x_2 + \text{..........} a_{mn} x_n \leq b_m$$

$$\text{and} \quad x_1 \geq 0, x_2 \geq 0, \text{............} x_n \geq 0$$

Example 1 (Slack Variables) (cont'd)

$$
\begin{aligned}
\text{minimize} \quad & \mathrm{c}_1 x_1 + \mathrm{c}_2 x_2 + \ldots\ldots\ldots \mathrm{c}_\mathrm{n} x_n \\
\text{subject to} \quad & a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots a_{1n} x_n + y_1 = b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots a_{2n} x_n + y_2 = b_2 \\
& . \\
& . \\
& . \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots a_{mn} x_n + y_m = b_m \\
\text{and} \quad & \mathrm{y}_1 \geq 0, y_2 \geq 0, \ldots\ldots\ldots\ldots y_m \geq 0 \\
\text{and} \quad & x_1 \geq 0, x_2 \geq 0, \ldots\ldots\ldots\ldots x_n \geq 0
\end{aligned}
$$

Example 2 (Free Variables)

- X_1 is free to take on either positive or negative values.
- We then write $X_1 = U_1 - V_1$, where $U_1 \geqq 0$ and $V_1 \geqq 0$.
- Substitute $U_1 - V_1$ for X_1, then the linearity of the constraints is preserved and all variables are now required to be nonnegative.

The Diet Problem

- We assume that there are available at the market *n* different foods and that the ith food sells at a price c_i per unit. In addition there are *m* basic nutritional ingredients and, to achieve a balanced diet, each individual must receive at least b_j units of the jth nutrient per day. Finally, we assume that each unit of food i contains a_{ji} units of the jth nutrient.

The Diet Problem (cont'd)

minimize $\mathrm{c}_1 x_1 + \mathrm{c}_2 x_2 + \ldots\ldots\ldots \mathrm{c}_n x_n$

subject to $a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots a_{1n}x_n \geq b_1$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots a_{2n}x_n \geq b_2$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots a_{mn}x_n \geq b_m$$

and $x_1 \geq 0, x_2 \geq 0, \ldots\ldots\ldots\ldots x_n \geq 0$

The Transportation Problem

Quantities a_1, a_2,…, a_m, respectively, of a certain product are to be shipped from each of m locations and received in amounts b_1, b_2,…, b_y, respectively, at each of n destinations. Associated with the shipping of a unit of product from origin i to destination j is a unit shipping cost c_{ij}. It is desired to determine the amounts x_{ij} to be shipped between each origin-destination pair (i=1,2,…,m; j=1,2,…,n)

The Transportation Problem (cont'd)

minimize $\sum_{ij} c_{ij} x_{ij}$

subject to $\sum_{j=1}^{n} x_{ij} = a_i$ for i=1,2,......,m

$\sum_{i=1}^{m} x_{ij} = b_j$ for j=1,2,......,n

$x_{ij} \geq 0$ for i=1,2,......m

j=1,2,......n