Introduction to Linear and Nonlinear Programming
Outline

- Introduction
  - types of problems
  - size of problems
  - iterative algorithms and convergence

- Basic properties of Linear Programs
  - introduction
  - examples of linear programming problems
Types of Problems

- Three parts
  - linear programming
  - unconstrained problems
  - constrained problems
- The last two parts comprise the subject of nonlinear programming
Linear Programming

- It is characterized by linear functions of the unknowns; the **objective** is linear in the unknowns, and the **constraints** are linear equalities or linear inequalities.

- Why are linear forms for objectives and constraints so popular in problem formulation?
  - A great number of constraints and objectives that arise in practice are indisputably linear (ex: budget constraint)
  - They are often the least difficult to define
Unconstrained Problems

- Are unconstrained problems devoid of structural properties as to preclude their applicability as useful models of meaningful problems?

- If the scope of a problem is broadened to the consideration of all relevant decision variables, there may then be no constraints.

- Many constrained problems are sometimes easily converted to unconstrained problems.
Constrained Problems

- Many complex problems cannot be directly treated in its entirety accounting for all possible choices, but instead must be decomposed into separate subproblems
- “Continuous variable programming”
Size of Problems

- Three classes of problems
  - small scale: five or fewer unknowns and constraints
  - intermediate scale: from five to a hundred variables
  - large scale: from a hundred to thousands variables
Iterative Algorithms and Convergence

- Most algorithms designed to solve large optimization problems are iterative.
- For LP problems, the generated sequence is of finite length, reaching the solution point exactly after a finite number of steps.
- For non-LP problems, the sequence generally does not ever exactly reach the solution point, but converges toward it.
Iterative Algorithms
and Convergence (cont’d)

- **Iterative algorithms**
  1. the creation of the algorithms themselves
  2. the verification that a given algorithm will in fact generate a sequence that converges to a solution
  3. the rate at which the generated sequence of points converges to the solution

- **Convergence rate theory**
  1. the theory is, for the most part, extremely simple in nature
  2. a large class of seemingly distinct algorithms turns out to have a common convergence rate
LP Problems’ Standard Form

minimize \[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \]

subject to \[ a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \]
\[ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \]
\[ \vdots \]
\[ \vdots \]
\[ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m \]

and \[ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \]
LP Problems’ Standard Form (cont’d)

\[
\min \ c^T x \\
\text{subject to} \ A x = b \quad \text{and} \ x \geq 0
\]

- Here \( x \) is an \( n \)-dimensional column vector,
  \( c^T \) is an \( n \)-dimensional row vector, \( A \) is an \( m \times n \) matrix, and \( b \) is an \( m \)-dimensional column vector.
Example 1 (Slack Variables)

minimize \( c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \)

subject to \( a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \)
\( a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \)
\( \vdots \)
\( a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m \)

and \( x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0 \)
Example 1 (Slack Variables) (cont’d)

minimize $c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + y_1 = b_1$
$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + y_2 = b_2$
. . .
$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + y_m = b_m$

and $y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0$

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$
Example 2 (Free Variables)

- $X_1$ is free to take on either positive or negative values.
- We then write $X_1 = U_1 - V_1$, where $U_1 \geq 0$ and $V_1 \geq 0$.
- Substitute $U_1 - V_1$ for $X_1$, then the linearity of the constraints is preserved and all variables are now required to be nonnegative.
The Diet Problem

- We assume that there are available at the market $n$ different foods and that the $i$th food sells at a price $C_i$ per unit. In addition there are $m$ basic nutritional ingredients and, to achieve a balanced diet, each individual must receive at least $b_j$ units of the $j$th nutrient per day. Finally, we assume that each unit of food $i$ contains $a_{ji}$ units of the $j$th nutrient.
The Diet Problem (cont’d)

minimize \[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \]
subject to \[ a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1 \]
\[ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \geq b_2 \]
\[ \vdots \]
\[ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \geq b_m \]
and \[ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \]
The Transportation Problem

Quantities $a_1, a_2, \ldots, a_m$, respectively, of a certain product are to be shipped from each of $m$ locations and received in amounts $b_1, b_2, \ldots, b_n$, respectively, at each of $n$ destinations. Associated with the shipping of a unit of product from origin $i$ to destination $j$ is a unit shipping cost $c_{ij}$. It is desired to determine the amounts $x_{ij}$ to be shipped between each origin-destination pair ($i=1,2,\ldots,m$; $j=1,2,\ldots,n$).
The Transportation Problem (cont’d)

minimize \( \sum_{ij} c_{ij} x_{ij} \)

subject to \( \sum_{j=1}^{n} x_{ij} = a_i \) for \( i=1,2,\ldots,m \)

\( \sum_{i=1}^{m} x_{ij} = b_j \) for \( j=1,2,\ldots,n \)

\( x_{ij} \geq 0 \) for \( i=1,2,\ldots,m \)

\( j=1,2,\ldots,n \)